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Extending equivariant maps into spaces with convex structure

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Abstract

We prove that if G is a compact Lie group, Y a G-space equipped with a topological local convex structure compatible with the action of G, then Y is a G-*ANE* for metrizable G-spaces. If, in addition, Y has a G-fixed point and admits a global convex structure compatible with the action of G, then Y is a G-*AE*. This is applied to show that certain hyperspaces related to the Banach–Mazur compacta are equivariant absolute extensors.

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1. Introduction

In the present paper we prove a general equivariant extension theorem for equivariant maps with values in G-spaces which possess a (local or global) topological convex structure compatible with the given action of a compact Lie group G. Our Theorems 3.2 and 3.3 extend essentially the existing versions of the equivariant Dugundji extension theorem (see [17,2–4]). At the same time these theorems are equivariant generalizations of some

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results in Himmelberg [16] and Curtis [14]. In Corollary 3.5 we extend these results to the case of proper actions of arbitrary (non-compact) Lie groups. Corollary 4.4 states that for a compact Lie group G, metrizable G-*ANE*'s are precisely those metrizable G-spaces that admit such a local G-convex structure. This characterization is especially useful when one considers hyperspaces of compact convex subsets of normed linear G-spaces. In this way we show that certain important spaces related to the famous Banach–Mazur compacta are equivariant absolute extensors. In conclusion, we also discuss some open questions.

2. Preliminaries

We refer to the monographs [13,22] for basic notions of the theory of G-spaces. However, below we recall some special definitions and facts that are necessary throughout the paper.

If G is a topological group and X is a G-space, for any $x \in X$ we denote the stabilizer (or the stationary subgroup) of x by $G_x = \{g \in G \mid gx = x\}$. For a subset $S \subset X$ and a subgroup $H \subset G$, $H(S)$ denotes the H-saturation of S, i.e., $H(S) = \{hs \mid h \in H,\ s \in S\}$. If $H(S) = S$, then we say that S is an H-invariant set. In particular, $G(x)$ denotes the G-orbit $\{gx \in X \mid g \in G\}$ of x. The orbit space is denoted by X/G. By G/H we will denote the G-space of cosets $\{gH \mid g \in G\}$ under the action induced by left translations.

A compatible metric ρ on a G-space X is called invariant or G-invariant, if $\rho(gx, gy) = \rho(x, y)$ for all $g \in G$ and $x, y \in X$.

We shall mean by a linear G-space, a real topological vector space L on which G acts continuously and linearly, i.e., $g(\lambda x + \mu y) = \lambda(gx) + \mu(gy)$ for every $g \in G$ and for all $x, y \in L$ and $\lambda, \mu \in \mathbb{R}$.

The terms "G-map" or "equivariant map" will include the continuity of the corresponding map.

A G-space Y is called an equivariant neighborhood extensor for a given G-space X (notation: $Y \in G\text{-}ANE(X)$), if for any closed invariant subset $A \subset X$ and any G-map $f : A \to Y$, there exist an invariant neighborhood U of A in X and a G-map $\psi : U \to Y$ that extends f. If, in addition, one can always take $U = X$, then we say that Y is an equivariant extensor for X (notation: $Y \in G\text{-}AE(X)$). The map ψ is called a G-extension of f.

If G is a compact group, then a G-space Y is called an equivariant absolute neighborhood extensor (notation: $Y \in G\text{-}ANE$), if $Y \in G\text{-}ANE(X)$ for any metrizable G-space X. Similarly, if $Y \in G\text{-}AE(X)$ for any metrizable G-space X, then Y is called an equivariant absolute extensor (notation: $Y \in G\text{-}AE$).

If G is the trivial group, we just get from here the definitions of the ordinary classes *ANE* and *AE*, respectively.

The notion of a slice is the key tool in our proofs; let us recall it:

Definition 2.1 [22]. Let G be a topological group, $H \subset G$ a closed subgroup and X a G-space. A subset $S \subset X$ is called an H-slice in X, if:

(1) S is H-invariant,
(2) the saturation $G(S)$ is open in X,

(3) if $g \in G \setminus H$, then $gS \cap S = \emptyset$,
(4) S is closed in $G(S)$.

The saturation $G(S)$ will be said to be a tubular set. If, in addition, $G(S) = X$, then we say that S is a global H-slice of X.

An open cover $\mathcal{U}$ of X is said to be tubular, if it consists exclusively of tubular sets.

Definition 2.2 [4]. Let G be a topological group, Z a G-space, U an open invariant subset of Z and $\{G(S_\mu)\}$ a tubular cover of U, where S_μ is an H_μ-slice in U. The cover $\{G(S_\mu)\}$ is called G-canonical with respect to Z, if:

(1) $\{G(S)\}$ is locally finite,
(2) for any index μ, there is a point $z \in U$ with $G_z = H_\mu$,
(3) for any point $a \in Z \setminus U$ and any neighborhood V_a of it in Z, there exists a neighborhood $W_a \subset V_a$ of a in Z such that if $g \in G$ and $gS_\mu \cap W_a \neq \emptyset$, then the following conditions hold:
   (a) $gS_\mu \subset V_a$,
   (b) there exists an element $h \in G$ such that $ha \in V_a$ and $H_\mu \subset g^{-1}G_{ha}g$.

Lemma 2.3 [4]. *If G is a compact Lie group and Z is a metrizable G-space, then every invariant open subset $U \subset Z$ admits a G-canonical cover with respect to Z.*

3. Extending equivariant maps

We begin with some auxiliary definitions.

Let G be a topological group and X a G-space. A cover $\mathcal{U}$ of X is said to be *a G-cover,* if $gU \in \mathcal{U}$ whenever $g \in G$ and $U \in \mathcal{U}$. For $\mathcal{U}$ an open G-cover of a G-space Y, and $k \geqslant 1$, let

$$Y^k(\mathcal{U}) = \bigcup_{U \in \mathcal{U}} U^k.$$

We always consider the diagonal G-action on Y^k. Then $Y^k(\mathcal{U})$ is an invariant open subspace of Y^k. Let Δ^k denote the standard k-dimensional simplex in Euclidean $(k+1)$-space $\mathbb{R}^{k+1}$, i.e.,

$$\Delta^k = \left\{ (t_1, \dots, t_{k+1}) \in \mathbb{R}^{k+1} \mid t_i \geqslant 0,\ \sum_{i=1}^{k+1} t_i = 1 \right\}.$$

Below for $(x_1, \dots, x_k) \in X^k$ we shall denote by $(x_1, \dots, \bar{\bar{x}}_m, \dots, x_k)$ the point $(x_1, \dots, x_{m-1}, x_{m+1}, \dots, x_k) \in X^{k-1}$.

The following definition is a straightforward equivariant analogue of the well-known notion of a local convex structure or an *LCS* space (see [16,14]).

Definition 3.1. A local G-convex structure for a G-space Y consists of an open G-cover $\mathcal{U}$ and a sequence of G-maps

$$h^k : Y^k(\mathcal{U}) \times \Delta^{k-1} \to Y, \quad k \geqslant 1$$

such that

(1) $h^k(y_1, \dots, y_k; t_1, \dots, t_k) = h^{k-1}(y_1, \dots, \bar{\bar{y}}_m, \dots, y_k; t_1, \dots, \bar{\bar{t}}_m, \dots, t_k)$ whenever $1 \leqslant m \leqslant k$ and $t_m = 0$,

(2) for every neighborhood N of any point $p \in Y$, there exists a neighborhood M of p such that $M^k \subset Y^k(\mathcal{U})$ and $h^k(M^k \times \Delta^{k-1}) \subset N$ for all $k \geqslant 1$.

Note that the second condition implies

(3) $h^k(y, \dots, y; t_1, \dots, t_k) = y$ for all $y \in Y$, $(t_1, \dots, t_k) \in \Delta^{k-1}$, $k = 1, 2, \dots$.

If $\mathcal{U} = \{Y\}$, then the local G-convex structure $(\mathcal{U}, (h^k))$ is called a global G-convex structure, or simply, a G-convex structure.

We shall say that Y is a *G-LCS* space (respectively, a *G-CS* space) whenever it admits a local G-convex structure (respectively, a G-convex structure).

A subset A of a *G-CS* space $(Y, (h^k))$ is called convex, if $h^k(a_1, \dots, a_k; t_1, \dots, t_k) \in A$ whenever $a_1, \dots, a_k \in A$ and $(t_1, \dots, t_k) \in \Delta^{k-1}$.

Evidently, a convex invariant subset of a *G-CS* space is also a *G-CS* space.

Here is our main result, which in the non-equivariant case was proved by Himmelberg [16] and Curtis [14]:

Theorem 3.2. *Let G be a compact Lie group. Then every G-LCS space is a G-ANE.*

Proof. Let $Y \in$ *G-LCS* and $(\mathcal{U}, (h^k))$ be a local G-convex structure on Y. Assume that $f : A \to Y$ is a G-map of a closed invariant subset A of a metrizable G-space X.

Fix an invariant metric ρ on X. For every orbit $G(a) \subset A$, there exists an invariant neighborhood Z_a of $G(a)$ in X such that $G_z \subset g^{-1}G_a g$ for each $z \in Z_a$ and some $g \in G$ (cf. [13, Chapter II, §5]). We denote by Z the union of all such Z_a. Then Z is an open invariant neighborhood of A in X.

Choose, by Lemma 2.3, an open tubular cover $\{G(S_\mu)\}$ of $Z \backslash A$, G-canonical with respect to Z. Recall that here S_μ is an H_μ-slice, where H_μ is a closed subgroup of G. Totally-order the index set $\{\mu\}$.

For every index μ, we denote by A_μ the H_μ-fixed point set of A, i.e.,

$$A_\mu = \{a \in A \mid ha = a \text{ for all } h \in H_\mu\}.$$

Since $H_\mu = G_z$ for some $z \in Z \setminus A$ and $G_z \subset G_a$ for some $a \in A$ (see Definition 2.2), we infer that $A_\mu \neq \emptyset$. Obviously, each A_μ is closed in Z. Since $\{G(S_\mu)\}$ is a G-canonical cover and f is a continuous map, for each $a \in A$ one can find neighborhoods $O_a(s) = \{z \in Z \mid \rho(z, a) < s\}$ and $T'_a \subset O_a(s)$ such that, if $gS_\mu \cap T'_a \neq \emptyset$, then the following three conditions are fulfilled:

$$gS_\mu \subset O_a(s), \tag{3.1}$$

there is an element $h \in G$ such that $ha \in O_a(s)$ and $H_\mu \subset g^{-1}G_{ha}g$, (3.2)

$$f\big(O_a(5s) \cap A\big) \subset U \quad \text{for some } U \in \mathcal{U}. \tag{3.3}$$

Let W be an open invariant neighborhood of A contained in $\bigcup\{T'_a \mid a \in A\}$. Set $T_a = W \cap T'_a$; then $W = \bigcup\{T_a \mid a \in A\}$. Set $Q_\mu = S_\mu \cap W$. Then Q_μ is an H_μ-slice in $W \setminus A$ and $\{G(Q_\mu)\}$ is a tubular cover of $W \setminus A$ satisfying the first and the third conditions of Definition 2.2 with respect to W.

Choose a partition of unity $\{\varphi_\mu\}$ consisting of invariant functions $\varphi_\mu : W \to [0,1]$ that is subordinated to the cover $\{G(Q_\mu)\}$, i.e., $\varphi_\mu^{-1}((0,1]) \subset G(Q_\mu)$ (see e.g., [4]).

For each H_μ-slice Q_μ, we select a point $x_\mu \in Q_\mu$ and associate a point $a_\mu \in A_\mu$ with it such that the following inequality is satisfied:

$$\rho(x_\mu, a_\mu) < 2\rho(x_\mu, A_\mu). \tag{3.4}$$

For every index μ, there is a unique G-map $\tilde{g}_\mu : G(Q_\mu) \to G/H_\mu$ such that

$$\begin{cases} z \in gQ_\mu \text{ for each } z \in G(Q_\mu) \text{ and each} \\ \text{representative } g \in \tilde{g}_\mu(z), \text{ i.e., } gH_\mu = \tilde{g}_\mu(z). \end{cases} \tag{3.5}$$

Furthermore, the map $G/H_\mu \times A_\mu \to A$ that associates to every pair $(gH_\mu, x) \in G/H_\mu \times A_\mu$, the point $gx \in A$, is well defined and continuous (see [4, Lemmas 3 and 4]).

Observe that, if $z \in W \setminus A$, then by local finiteness of the cover $\{G(Q_\mu)\}$, there exists only a finite number of indices $\mu_1, \ldots, \mu_k$ such that $z \in \bigcap_{i=1}^k G(Q_{\mu_i})$ and $\varphi_\mu(z) = 0$ for all $\mu \notin \{\mu_1, \ldots, \mu_k\}$.

Let $\mu \in \{\mu_1, \ldots, \mu_k\}$ and $g_\mu \in \tilde{g}_\mu(z)$. Then $z \in T_a$ for some $a \in A$ and by (3.5), $z \in g_\mu Q_\mu$. Thus $z \in g_\mu Q_\mu \cap T_a$, implying $g_\mu S_\mu \cap T'_a \neq \emptyset$. Then, it follows from (3.1) that $g_\mu Q_\mu \subset g_\mu S_\mu \subset O_a(s)$, and hence, $g_\mu x_\mu \in O_a(s)$.

Let $h \in G$ be as in (3.2). Then $ha \in O_a(s)$ and $H_\mu \subset g_\mu^{-1}G_{ha}g_\mu$, and hence, $g_\mu^{-1}ha \in A_\mu$.

Next, we have

$$\begin{cases} \rho(a, g_\mu a_\mu) \leqslant \rho(a, g_\mu x_\mu) + \rho(g_\mu x_\mu, g_\mu a_\mu) < s + \rho(x_\mu, a_\mu) \\ \quad < s + 2\rho(x_\mu, A_\mu) \leqslant s + 2\rho\big(x_\mu, g_\mu^{-1}ha\big) = s + 2\rho(g_\mu x_\mu, ha) < s + 4s = 5s. \end{cases} \tag{3.6}$$

(Here we used the G-invariance of the metric ρ, the inequality (3.4) and the inclusions $g_\mu x_\mu, \in O_a(s)$, $ha \in O_a(s)$ and $g_\mu^{-1}ha \in A_\mu$.)

Thus, for each $\mu \in \{\mu_1, \ldots, \mu_k\}$, the point $\tilde{g}_\mu(z)a_\mu = g_\mu a_\mu$ belongs to $O_a(5s) \cap A$. Then (3.3) yields that the points

$$f\big(\tilde{g}_{\mu_1}(z)a_{\mu_1}\big), \ldots, f\big(\tilde{g}_{\mu_k}(z)a_{\mu_k}\big)$$

belong to a set $U \in \mathcal{U}$, so

$$h^k\big(f\big(\tilde{g}_{\mu_1}(z)a_{\mu_1}\big), \ldots, f\big(\tilde{g}_{\mu_k}(z)a_{\mu_k}\big);\ \varphi_{\mu_1}(z), \ldots, \varphi_{\mu_k}(z)\big)$$

is a well-defined point of Y, where $\mu_i < \mu_j$ if $i < j$.

Consequently, setting

$$f'(z) = \begin{cases} f(z), & \text{if } z \in A \\ h^k\big(f\big(\tilde{g}_{\mu_1}(z)a_{\mu_1}\big), \ldots, f\big(\tilde{g}_{\mu_k}(z)a_{\mu_k}\big);\ \varphi_{\mu_1}(z), \ldots, \varphi_{\mu_k}(z)\big), \\ \qquad \text{if } z \in W \setminus A, \end{cases}$$

we get a well-defined map $f' : W \to Y$ that extends f.

We claim that f' is continuous. First, let us check its continuity on $W \setminus A$. For, let $z_0 \in W \setminus A$ be arbitrary. Using local finiteness of the cover $\{G(Q_\mu)\}$, we take a neighborhood T of z_0 in $W \setminus A$ with the property that only for a finite number of indices $\mu_1, \ldots, \mu_m$, the intersection $G(Q_{\mu_i}) \cap T$ is nonempty, where $\mu_i < \mu_j$ if $i < j$. To avoid notational complications we assume, without loss of generality, that the first p indices $\mu_1, \ldots, \mu_p$, $p \leqslant m$, are the only indices such that $z_0 \in G(Q_{\mu_i})$.

If $p = m$, then for all $z \in T \cap \bigcap_{i=1}^m G(Q_{\mu_i})$, we have

$$f'(z) = h^m\big(f\big(\tilde{g}_{\mu_1}(z)a_{\mu_1}\big), \ldots, f\big(\tilde{g}_{\mu_m}(z)a_{\mu_m}\big);\ \varphi_{\mu_1}(z), \ldots, \varphi_{\mu_m}(z)\big).$$

Now continuity of f' on $T \cap \bigcap_{i=1}^m G(Q_{\mu_i})$, and hence at z_0, follows from continuity of the maps f, $\tilde{g}_{\mu_i}$, φ_{μ_i} and h^m.

Assume that $p < m$. Let L be a neighborhood of the point

$$f'(z_0) = h^p\big(f\big(\tilde{g}_{\mu_1}(z_0)a_{\mu_1}\big), \ldots, f\big(\tilde{g}_{\mu_p}(z_0)a_{\mu_p}\big);\ \varphi_{\mu_1}(z_0), \ldots, \varphi_{\mu_p}(z_0)\big).$$

Using the first condition of Definition 3.1 and continuity of h^m, one can prove the following elementary fact:

Claim. *Let $\{y_1, \ldots, y_p\} \subset Y^p(\mathcal{U})$ and $(t_1, \ldots, t_p) \in \Delta^{p-1}$. Assume that L is a neighborhood of $h^p(y_1, \ldots, y_p; t_1, \ldots, t_p)$. Then for any collection of compact sets $K_j \subset Y$, $p+1 \leqslant j \leqslant m$, there exist neighborhoods R_i of y_i, $1 \leqslant i \leqslant p$, and a number $\delta > 0$ such that*

$$h^m(z_1, \ldots, z_p, z_{p+1}, \ldots, z_m; \tau_1, \ldots, \tau_p, \tau_{p+1}, \ldots, \tau_m) \in L$$

whenever

(1) $z_i \in R_i$ *and* $|\tau_i - t_i| < \delta$ *for* $i = 1, \ldots, p$;
(2) $z_j \in K_j$ *and* $\tau_j < \delta$ *for* $j = p+1, \ldots, m$.

In our case $y_i = f(\tilde{g}_{\mu_i}(z_0)a_{\mu_i})$, $t_i = \varphi_{\mu_i}(z_0)$ for $1 \leqslant i \leqslant p$ and $K_j = f(G(a_{\mu_j}))$ for $p+1 \leqslant j \leqslant m$. Choose a real $\delta > 0$ and a neighborhood R_i of $f(\tilde{g}_{\mu_i}(z_0)a_{\mu_i})$, $1 \leqslant i \leqslant p$, according to the above claim.

By continuity of the maps f, $\tilde{g}_{\mu_i}$ and φ_{μ_i}, one can choose a neighborhood $E \subset \bigcap_{i=1}^p G(Q_{\mu_i})$ of z_0 such that

$$\begin{cases} f\big(\tilde{g}_{\mu_i}(z)a_{\mu_i}\big) \in R_i,\ \big|\varphi_{\mu_i}(z) - \varphi_{\mu_i}(z_0)\big| < \delta & \text{for all } z \in E,\ i = 1, \ldots, p \\ \text{and } \varphi_{\mu_j}(z) < \delta & \text{for all } z \in E,\ j = p+1, \ldots, m. \end{cases}$$

Then, it follows from the above claim that

$$h^m\big(f\big(\tilde{g}_{\mu_1}(z)a_{\mu_1}\big), \ldots, f\big(\tilde{g}_{\mu_p}(z)a_{\mu_p}\big), z_{p+1}, \ldots, z_m;\ \varphi_{\mu_1}(z), \ldots, \varphi_{\mu_m}(z)\big) \in L \tag{3.7}$$

for all $z \in E$ and $z_j \in f(G(a_{\mu_j}))$, $p+1 \leqslant j \leqslant m$.

We claim that $f'(z) \in L$ for all $z \in E$. Indeed, let $z \in E$ be arbitrary. Again, for simplicity and without loss of generality, we assume that $\mu_1, \dots, \mu_n$, $p \leqslant n \leqslant m$, are all the indices such that $z \in G(Q_{\mu_i})$. Then,

$$f'(z) = h^n\big(f\big(\tilde{g}_{\mu_1}(z)a_{\mu_1}\big), \dots, f\big(\tilde{g}_{\mu_n}(z)a_{\mu_n}\big);\ \varphi_{\mu_1}(z), \dots, \varphi_{\mu_n}(z)\big).$$

If $n = m$, then since $f(\tilde{g}_{\mu_j}(z)a_{\mu_j}) \in f(G(a_{\mu_j}))$ for $p+1 \leqslant j \leqslant m$, it follows from (3.7) that $f(z) \in L$.

If $n < m$, then $\varphi_{\mu_j}(z) = 0$ for $n+1 \leqslant j \leqslant m$, and according to the first condition of Definition 3.1, we have

$$\begin{aligned} f'(z) &= h^n\big(f\big(\tilde{g}_{\mu_1}(z)a_{\mu_1}\big), \dots, f\big(\tilde{g}_{\mu_n}(z)a_{\mu_n}\big);\ \varphi_{\mu_1}(z), \dots, \varphi_{\mu_n}(z)\big) \\ &= h^m\big(f\big(\tilde{g}_{\mu_1}(z)a_{\mu_1}\big), \dots, f\big(\tilde{g}_{\mu_n}(z)a_{\mu_n}\big), f(a_{\mu_{n+1}}), \dots, f(a_{\mu_m}); \\ &\quad \varphi_{\mu_1}(z), \dots, \varphi_{\mu_m}(z)\big). \end{aligned}$$

Now, since $f(\tilde{g}_{\mu_j}(z)a_{\mu_j}) \in f(G(a_{\mu_j}))$ for $p+1 \leqslant j \leqslant n$, and $f(a_{\mu_j}) \in f(G(a_{\mu_j}))$ for $n+1 \leqslant j \leqslant m$, again it follows from (3.7) that $f'(z) \in L$. Thus, f' is continuous at the point z_0.

It remains only to verify the continuity of f' on A. For, let $a \in A$ and let N be a neighborhood of $f'(a) = f(a)$ in Y. Choose a neighborhood M of $f'(a)$ satisfying the second condition of Definition 3.1. As above, we denote by $O_a(r)$ the open r-ball in W centered at the point a. Since f is continuous, there exists an $O_a(\varepsilon)$ such that $f(A \cap O_a(\varepsilon)) \subset M$. Let $V_a = O_a(\varepsilon/6)$. By the choice of the cover $\{G(Q_\mu)\}$, there exists a neighborhood $W_a \subset V_a$ satisfying the third condition of Definition 2.2.

We assert that $f'(W_a) \subset N$. In fact, if $z \in W_a \cap A$, then $f'(z) = f(z) \in M \subset N$. If $z \in W_a \cap (W \setminus A)$, then there exists only a finite number of indices $\mu_1, \dots, \mu_k$ with $\mu_i < \mu_j$ if $i < j$, such that $z \in \bigcap_{i=1}^k G(Q_{\mu_i})$ and $\varphi_\mu(z) = 0$ for all $\mu \neq \{\mu_1, \dots, \mu_k\}$.

Then

$$f'(z) = h^k\big(f\big(\tilde{g}_{\mu_1}(z)a_{\mu_1}\big), \dots, f\big(\tilde{g}_{\mu_k}(z)a_{\mu_k}\big);\ \varphi_{\mu_1}(z), \dots, \varphi_{\mu_k}(z)\big).$$

Let $\mu \in \{\mu_1, \dots, \mu_k\}$ and $g_\mu \in \tilde{g}_\mu(z)$. By (3.5), $z \in g_\mu Q_\mu$, so $z \in g_\mu Q_\mu \cap W_a$. By the choice of W_a, it then follows that $g_\mu Q_\mu \subset V_a$, in particular, $g_\mu x_\mu \in V_a = O_a(\varepsilon/6)$. It then follows that

$$\rho(a, g_\mu a_\mu) \leqslant \rho(a, g_\mu x_\mu) + \rho(g_\mu x_\mu, g_\mu a_\mu) < \varepsilon/6 + \rho(x_\mu, a_\mu) \tag{3.8}$$

(here we used the G-invariance of the metric ρ).

Since $g_\mu Q_\mu \cap W_a \neq \emptyset$, by the choice of W_a (see the third condition of Definition 2.2), we have $H_\mu \subset g_\mu^{-1} G_{ha} g_\mu$ for some $h \in G$ with $ha \in V_a$. But

$$g_\mu^{-1} G_{ha} g_\mu = G_{g_\mu^{-1}ha}, \quad \text{so } H_\mu \subset G_{g_\mu^{-1}ha}, \quad \text{i.e.,} \quad g_\mu^{-1}ha \in A_\mu.$$

Then using (3.4), the inclusions $g_\mu x_\mu$, $ha \in O_a(\varepsilon/6)$ and the G-invariance of ρ, we get

$$\rho(x_\mu, a_\mu) < 2\rho(x_\mu, A_\mu) \leqslant 2\rho\big(x_\mu, g_\mu^{-1}ha\big) = 2\rho(g_\mu x_\mu, ha) < 2\varepsilon/3. \tag{3.9}$$

Now, (3.8) and (3.9) yield

$$\rho(a, g_\mu a_\mu) < \varepsilon/6 + 2\varepsilon/3 < \varepsilon.$$

Hence, $g_\mu a_\mu \in O_a(\varepsilon) \cap A$ for each $\mu \in \{\mu_1, \dots, \mu_k\}$. But then $f(\tilde{g}_\mu(z)a_\mu) = f(g_\mu a_\mu) \in M$, and by the choice of M, it then follows that

$$f'(z) = h^k\bigl(f\bigl(\tilde{g}_{\mu_1}(z)a_{\mu_1}\bigr), \dots, f\bigl(\tilde{g}_{\mu_k}(z)a_{\mu_k}\bigr);\ \varphi_{\mu_1}(z), \dots, \varphi_{\mu_k}(z)\bigr) \in N.$$

Thus $f'(W_a) \subset N$, and the continuity of f' is proved. Furthermore, f' is equivariant: this is immediate from the equivariance of the maps f, $\tilde{g}_\mu$, h^k, and from the invariance of the functions φ_μ. This completes the proof. □

Theorem 3.3. *Let G be a compact Lie group. Then every G-CS space with a G-fixed point is a G-AE.*

Proof. Let Y be a G-*CS* space with the global G-convex structure (h^k) and let $y_0 \in Y$ be a G-fixed point. Then by Theorem 3.2, $Y \in G$-*ANE*. On the other hand, Y is G-contractible to the point y_0. Indeed, the map $F : Y \times [0,1] \to Y$ defined by $F(y,t) = h^2(y, y_0; 1-t, t)$ is a G-map, and it follows from the first and third conditions of Definition 3.1 that $F(y,0) = y$ and $F(y,1) = y_0$. Thus, Y is G-contractible. It remains only to observe that a G-contractible G-*ANE* is a G-*AE* (see e.g., [11, Lemma 4.1]). □

Remark 3.4. For metrizable G-spaces the converse of Theorems 3.2 and 3.3 is also true; see Corollary 4.4 below.

Theorems 3.2 and 3.3 can easily be extended to the case of *proper* actions of non-compact Lie groups. To do so, we need first to recall some more definitions.

Following Palais [23, Definition 1.2.2], we call a G-space X proper, if:

(1) G is a locally compact Hausdorff topological group,
(2) X is completely regular Hausdorff space,
(3) every point of X has a neighborhood V such that for every point of X, there is a neighborhood U with the property that the set $\langle U, V\rangle = \{g \in G \mid gU \cap V \neq \emptyset\}$ has compact closure in G.

Clearly, if G is compact, every G-space is proper.

By $\mathcal{M}$ we denote the class of all metrizable proper G-spaces X that admit a G-invariant metric. Palais [23] proved that $\mathcal{M}$ includes all separable metrizable proper G-spaces. The question whether $\mathcal{M}$ coincides with the class of all metrizable proper G-spaces still remains open.

Below we will denote by G-*AE*$(\mathcal{M})$ (respectively, G-*ANE*$(\mathcal{M})$) the class of all G-spaces that are G-equivariant (respectively, neighborhood) extensors for each G-space M belonging to $\mathcal{M}$.

Corollary 3.5. *Let G be an arbitrary Lie group. Then*

(1) *Every G-LCS space is a G-ANE$(\mathcal{M})$.*
(2) *Every G-CS space X, with the property that for each compact subgroup $H \subset G$ there is an H-fixed point in X, is a G-AE$(\mathcal{M})$.*

This corollary is a simple combination of Theorem 3.2 (local case) and Theorem 3.3 (global case) with the following result by Abels [2, Theorem 4.4] (see also [9, Theorem 5]), which reduces the general case to that of compact group actions:

Theorem 3.6. *Let G be a locally compact group and Γ be a collection of compact subgroups of G such that every compact subgroup of G is conjugate to a subgroup $K \in \Gamma$. Let X be a G-space which is a K-ANE (respectively, a K-AE) for each $K \in \Gamma$. Then X is a G-ANE$(\mathcal{M})$ (respectively, a G-AE$(\mathcal{M})$).*

4. Applications

In this section we shall consider several important examples of G-*LCS* spaces which are related to the Banach–Mazur compacta. The above results are applied to establish some useful extensorial properties of these spaces.

Example 4.1. Let G be a topological group, L a locally convex linear G-space and Y a convex invariant subset of L. Then the maps $h^k : Y^k \times \Delta^{k-1} \to Y$, $k \geqslant 1$, defined by

$$h^k(y_1, \dots, y_k; t_1, \dots, t_k) = \sum_{i=1}^{k} t_i y_i,$$

determine a G-convex structure on Y.

Example 4.2. Let $(\mathcal{U}, (h^k))$ be a local G-convex structure on a G-space Y and let $r : Y \to A$ be a G-retraction on an invariant subset $A \subset Y$.

Then $\mathcal{U}_A = \{U \cap A \mid U \in \mathcal{U}\}$ is an open G-cover of A. As above, for each $k \geqslant 1$, we denote

$$A^k(\mathcal{U}_A) = \bigcup_{U \in \mathcal{U}} (U \cap A)^k.$$

Define the maps $l^k : A^k(\mathcal{U}_A) \times \Delta^{k-1} \to A$, by

$$l^k(a_1, \dots, a_k; t_1, \dots, t_k) = r\big(h^k(a_1, \dots, a_k; t_1, \dots, t_k)\big).$$

It is easy to see that $(\mathcal{U}_A, (l^k))$ determine a local G-convex structure on A.

Example 4.3. Every open invariant subset of a G-*LCS* space is a G-*LCS* space.

Indeed, let X be a G-*LCS* space and $(\mathcal{U}, (h^k))$ a local G-convex structure on it. Assume that Y is an invariant open subset of X. By virtue of the second condition of Definition 3.1, for every $y \in Y$, there is a neighborhood $O_y \subset Y$ such that $O_y^k \subset X^k(\mathcal{U})$ and $h^k(O_y^k \times \Delta^{k-1}) \subset Y$ for all $k \geqslant 1$. Let $\mathcal{V}$ be the open G-cover of Y consisting of all the sets gO_y, $g \in G$, $y \in Y$.

For every $k \geqslant 1$, define the map $l^k : Y^k(\mathcal{V}) \times \Delta^{k-1} \to Y$ to be the restrictions of the map $h^k : X^k(\mathcal{U}) \times \Delta^{k-1} \to X$. Then it is easy to check that $(\mathcal{V}, (l^k))$ is a local G-convex structure on Y.

It is known [5, Theorem 2 or Theorem 3] that every metrizable G-space Y, with G a compact Hausdorff group, can be embedded in a normed linear G-space L as a closed invariant subset. If Y is a metrizable G-ANE, then there is a G-retraction $r : U \to Y$, where U is an open invariant neighborhood of Y in L. It then follows from Examples 4.1–4.3 that Y is a G-LCS space. If Y is a G-AE, then one can take $U = L$; so in this case Y is a G-CS space and the point $r(0)$ is a G-fixed point in Y, where 0 stands for the origin of L. The converse follows from Theorems 3.2 and 3.3. So, we have the following useful characterization of the metrizable G-ANE's and G-AE's:

Corollary 4.4. *Let G be a compact Lie group and Y a metrizable G-space. Then Y is a G-ANE* (*respectively, a G-AE*) *iff Y is a G-LCS space* (*respectively, Y is a G-CS space with a G-fixed point*).

Let G be a topological group and $(Z, \|\cdot\|)$ a normed linear G-space (recall that G acts by means of linear operators on Z). Let X be a convex invariant subset of Z. Denote by $cc(X)$ the hyperspace of all nonempty compact convex subsets of X equipped with the Hausdorff metric:

$$d_H(A, B) = \max\Big\{\sup_{b\in B} \operatorname{dist}(b, A), \sup_{a\in A} \operatorname{dist}(a, B)\Big\} \quad \text{for } A, B \in cc(X).$$

The topology defined by d_H on $cc(X)$ is an invariant of the topology of X.

We shall consider the induced action of G on $cc(X)$ defined as follows:

$$(g, A) \mapsto gA; \quad gA = \{ga \mid a \in A\}, \quad \text{for all } g \in G,\ A \in cc(X).$$

It is easy to see that this action is continuous, so $cc(X)$ is a G-space.

A great deal of [21, Chapter XVIII] is devoted to "*cc*-hyperspaces". Here we are interested in a G-convex structure on $cc(X)$.

Consider the maps

$$h^k : cc(X)^k \times \Delta^{k-1} \to cc(X), \quad k = 1, 2, \ldots,$$

defined as follows:

$$h^k(A_1, \ldots, A_k; t_1, \ldots, t_k) = t_1A_1 + \cdots + t_kA_k = \left\{\sum_{i=1}^{k} t_ia_i \mid a_i \in A_i\right\},$$

the Minkowski linear combination.

Theorem 4.5. *For any topological group G, the above defined maps* (h^k) *determine a G-convex structure on $cc(X)$.*

Proof. Clearly, $h^k(A_1, \ldots, A_k; t_1, \ldots, t_k)$ is a compact convex subset of Z, and by convexity of X, it lies in X.

The first condition of Definition 3.1 is evident.

Let us verify continuity of h^k. Let $(t_1^0, \ldots, t_k^0) \in \Delta^{k-1}$ and $A_1^0, \ldots, A_k^0 \in cc(X)$ be fixed. Assume that $1 > \varepsilon > 0$. Since the sets $A_i^0 \subset Z$ are bounded, there is a real $M > 1$ such that $\|a_i^0\| \leqslant M$ for all $a_i^0 \in A_i^0$, $i = 1, \ldots, k$. Choose $\delta = \varepsilon/4Mk$.

We claim that, if $A_i \in cc(X)$, $i = 1, \dots, k$ and $(t_1, \dots, t_k) \in \Delta^{k-1}$ with

$$d_H\big(A_i, A_i^0\big) < \delta \quad \text{and} \quad \big|t_i - t_i^0\big| < \delta,$$

then

$$d_H\big(h^k(A_1, \dots, A_k; t_1, \dots, t_k), h^k\big(A_1^0, \dots, A_k^0; t_1^0, \dots, t_k^0\big)\big) < \varepsilon. \tag{4.1}$$

Indeed, let $a_i \in A_i$ and $|t_i - t_i^0| < \delta$. Then there are $a_i^0 \in A_i^0$ such that $\|a_i - a_i^0\| < \delta$. It then follows that

$$\begin{aligned}
\left\| \sum_{i=1}^k t_i a_i - \sum_{i=1}^k t_i^0 a_i^0 \right\| &\leqslant \left\| \sum_{i=1}^k t_i a_i - \sum_{i=1}^k t_i a_i^0 \right\| + \left\| \sum_{i=1}^k t_i a_i^0 - \sum_{i=1}^k t_i^0 a_i^0 \right\| \\
&\leqslant \sum_{i=1}^k t_i \big\|a_i - a_i^0\big\| + \sum_{i=1}^k \big|t_i - t_i^0\big| \big\|a_i^0\big\| < \sum_{i=1}^k \delta t_i + \sum_{i=1}^k \delta \big\|a_i^0\big\| \\
&\leqslant \delta + \delta k M \leqslant \varepsilon/4 + \varepsilon/4 = \varepsilon/2.
\end{aligned}$$

Thus,

$$\operatorname{dist}\left(\sum_{i=1}^k t_i a_i, h^k\big(A_1^0, \dots, A_k^0; t_1^0, \dots, t_k^0\big) \right) \leqslant \varepsilon/2. \tag{4.2}$$

Analogously, it can be shown that for every point $\sum_{i=1}^k t_i^0 a_i^0 \in h^k(A_1^0, \dots, A_k^0; t_1^0, \dots, t_k^0)$, the following inequality holds:

$$\operatorname{dist}\left(\sum_{i=1}^k t_i^0 a_i^0, h^k(A_1, \dots, A_k; t_1, \dots, t_k) \right) \leqslant \varepsilon/2. \tag{4.3}$$

Now (4.1) is immediate from (4.2) and (4.3), which proves the continuity of the function h^k, $k = 1, 2, \dots$.

Let us check the second condition of Definition 3.1. Let $P \in cc(X)$. We have to find for any $\varepsilon > 0$, a number $\delta > 0$ such that, if

$$A_i \in cc(X) \quad \text{and} \quad d_H(P, A_i) < \delta, \quad i = 1, \dots, k, \tag{4.4}$$

then

$$d_H\big(P, h^k(A_1, \dots, A_k; t_1, \dots, t_k)\big) < \varepsilon \tag{4.5}$$

for all $(t_1, \dots, t_k) \in \Delta^{k-1}$ and $k = 1, 2, \dots$.

It turns out that one can take $\delta = \varepsilon$. Indeed, let $p \in P$ be arbitrary. If (4.4) holds, then for every $1 \leqslant i \leqslant k$, there exists some $a_i \in A_i$ such that $\|p - a_i\| < \delta$. By convexity of balls in Z, one has $\|p - \sum_{i=1}^k t_i a_i\| < \delta$ for all $(t_1, \dots, t_k) \in \Delta^{k-1}$. Consequently,

$$\operatorname{dist}\big(p, h^k(A_1, \dots, A_k; t_1, \dots, t_k)\big) < \delta \tag{4.6}$$

for all $p \in P$ and $(t_1, \dots, t_k) \in \Delta^{k-1}$.

Analogously, it follows from (4.4) that for any point $a_j \in A_j$, $j = 1, \dots, k$, there is a point $p_j \in P$ such that $\|p_j - a_j\| < \delta$. But then,

$$\left\| \sum_{j=1}^{k} t_j a_j - \sum_{j=1}^{k} t_j p_j \right\| = \left\| \sum_{j=1}^{k} t_j (a_j - p_j) \right\| \leqslant \sum_{j=1}^{k} t_j \|a_j - p_j\| < \delta \sum_{j=1}^{k} t_j = \delta. \tag{4.7}$$

Because of convexity, the point $\sum_{j=1}^{k} t_j p_j$ belongs to P, and hence, (4.7) implies

$$\operatorname{dist}\left(\sum_{j=1}^{k} t_j a_j, P \right) < \delta \tag{4.8}$$

for all $(a_1, \ldots, a_k) \in A_1 \times \cdots \times A_k$ and $(t_1, \ldots, t_k) \in \Delta^{k-1}$. Now (4.5) immediately follows from the definition of the Hausdorff metric and from (4.6) and (4.8). This completes the proof. $\square$

Corollary 4.6. *Let G be a Lie group and X be a convex invariant subset of a normed linear G-space. Then $cc(X)$ is a G-ANE$(\mathcal{M})$. If in addition X is complete, then $cc(X) \in G$-AE$(\mathcal{M})$.*

Proof. The first claim is immediate from Corollary 3.5(1) and Theorem 4.5. The second one follows from Corollary 3.5(2) and Theorem 4.5, if we observe that for any compact subgroup $H \subset G$ and any point $x \in X$, the closed convex hull $\overline{\operatorname{conv} H(x)}$ belongs to $cc(X)$ (see e.g., [24, Theorem 3.20(c)]) and is an H-fixed point. $\square$

Below, as usual, we denote by $GL(n)$ the full linear group, by $O(n)$—the orthogonal group and by B^n—the ordinary Euclidean unit ball:

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leqslant 1\}.$$

Example 4.7. By Theorem 4.5, the maps $h^k : (cc(\mathbb{R}^n))^k \times \Delta^{k-1} \to cc(\mathbb{R}^n)$, $k \geqslant 1$, defined by $h^k(A_1, \ldots, A_k; t_1, \ldots, t_k) = t_1 A_1 + \cdots + t_n A_k$, the Minkowski linear combination, determine a $GL(n)$-convex structure on $cc(\mathbb{R}^n)$.

It is easy to see that the following subsets of $cc(\mathbb{R}^n)$ are $GL(n)$-invariant and convex with respect to the $GL(n)$-convex structure above defined:

(1) $cb(\mathbb{R}^n) = \{A \in cc(\mathbb{R}^n) \mid \operatorname{Int} A \neq \emptyset\}$—all compact convex bodies.
(2) $\mathcal{B}(n) = \{A \in cb(\mathbb{R}^n) \mid A = -A\}$—all compact convex centrally symmetric bodies.

Thus, $cc(\mathbb{R}^n)$, $cb(\mathbb{R}^n)$ and $\mathcal{B}(n)$ are $GL(n)$-*CS* spaces.

The sets $cc(B^n)$, $cc(B^n) \cap cb(\mathbb{R}^n)$ and $cc(B^n) \cap \mathcal{B}(n)$ are $O(n)$-invariant and convex subsets of $cc(\mathbb{R}^n)$; so all these sets are $O(n)$-*CS* spaces.

Let $\mathcal{N}(n)$ be the $GL(n)$-space of all norms $\varphi : \mathbb{R}^n \to \mathbb{R}$ endowed with the compact-open topology and with the $GL(n)$-action $GL(n) \times \mathcal{N}(n) \to \mathcal{N}(n)$ defined by $(g\varphi)(x) = \varphi(g^{-1}x)$, where $g \in GL(n)$, $\varphi \in \mathcal{N}(n)$ and $x \in \mathbb{R}^n$. In [8] it was observed that $\mathcal{B}(n)$ is just $GL(n)$-equivariantly homeomorphic (by means of the Minkowski functional) to $\mathcal{N}(n)$.

In [8] it was also established that $\mathcal{N}(n)$, and hence, $\mathcal{B}(n)$ is a proper $GL(n)$-space. Furthermore, $\mathcal{N}(n)$ is $GL(n)$-$AE(\mathcal{M})$ as it was noticed in [7, Proof of Corollary 7]. Thus, $\mathcal{B}(n) \in GL(n)$-AE.

Below we show that the same inclusion $\mathcal{B}(n) \in GL(n)$-AE follows more directly from our results above, which are applicable in more general situations also.

Corollary 4.8.

(1) *The $O(n)$-spaces $cc(\mathbb{R}^n)$, $cb(\mathbb{R}^n)$, $\mathcal{B}(n)$, $cc(B^n)$, $cc(B^n) \cap cb(\mathbb{R}^n)$ and $cc(B^n) \cap \mathcal{B}(n)$ all are $O(n)$-AE's.*
(2) *The $GL(n)$-spaces $cc(\mathbb{R}^n)$, $cb(\mathbb{R}^n)$ and $\mathcal{B}(n)$ all are $GL(n)$-$AE(\mathcal{M})$'s.*

Proof. (1) We shall prove the first claim only for $cc(\mathbb{R}^n)$, the other cases are quite similar.

Since $cc(\mathbb{R}^n)$ is an $O(n)$-CS space and the unit ball B^n is an $O(n)$-fixed point in $cc(\mathbb{R}^n)$, it then follows from Theorem 3.3 that $cc(\mathbb{R}^n)$ is an $O(n)$-AE.

(2) It is well known that the orthogonal group $O(n)$ is a maximal compact subgroup of $GL(n)$, i.e., each compact subgroup of $GL(n)$ is conjugate to a subgroup of $O(n)$. Now the second claim follows from the first one and Theorem 3.6. □

The $GL(n)$-space $\mathcal{B}(n)$ is of a special interest in Geometric theory of finite-dimensional Banach spaces [19, §5.2], as well as in Convex geometry [15]. The orbit space $\mathcal{B}(n)/GL(n)$ is just the famous Banach–Mazur compactum $BM(n)$ and it is an interesting topological object (see [19, p. 1191] and [25, Problem 899]).

A classical result by John [18] asserts that each compact convex body in $\mathbb{R}^n$ has a unique minimal (respectively, maximal) volume ellipsoid containing (respectively, contained in) the body. Let $L(n)$ (respectively, $J(n)$) denote the set of all bodies $A \in \mathcal{B}(n)$ for which the ordinary Euclidean unit ball B^n is the minimal (respectively, the maximal) volume ellipsoid. Clearly, $L(n)$ and $J(n)$ are $O(n)$-invariant subsets of $\mathcal{B}(n)$. But, as it is easy to see, neither $L(n)$ nor $J(n)$ is convex with respect to the convex structure of $\mathcal{B}(n)$ defined in Example 4.7. Therefore, at this stage, one cannot apply Theorem 3.2 to deduce that $L(n)$ and $J(n)$ are $O(n)$-ANE's.

However, in [10] it was proved that $L(n)$ and $J(n)$ are global $O(n)$-slices for the $GL(n)$-space $\mathcal{B}(n)$. In combination with a result of Abels [1, Theorem 1.2], this yields that there exists an $O(n)$-equivariant retraction $r : \mathcal{B}(n) \to L(n)$ such that for every $A \in \mathcal{B}(n)$, $r(A)$ belongs to the $GL(n)$-orbit $GL(n)(A)$. Hence, together with Corollary 4.8(1), this yields the following

Corollary 4.9 [10]. *The $O(n)$-spaces $L(n)$ and $J(n)$ are $O(n)$-AE's.*

Besides, the three orbit spaces $\mathcal{B}(n)/GL(n)$, $L(n)/O(n)$ and $J(n)/O(n)$ are just homeomorphic to each other (see [10, Corollary 1 and Remark 1]).

On the other hand, $L(n)$ is directly linked to $cc(B^n)$, and both hyperspaces of the closed unit ball B^n have several similar properties. Namely, for $n \geqslant 2$, $cc(B^n)$ is homeomorphic to the Hilbert cube [21, Theorem 18.4], and the same is true for $L(n)$ [12, Theorem 4]. Furthermore, by Corollary 4.8(1), $cc(B^n)$ is an $O(n)$-AE, and by Corollary 4.9, the same is true for $L(n)$. Consequently, it immediately follows from [6, Theorem 8] that the orbit spaces $L(n)/O(n)$ and $cc(B^n)/O(n)$ are *AE*'s. However, $L(n)$ and $cc(B^n)$ are different as $O(n)$-spaces. It suffices to observe that in $L(n)$ there is only one $O(n)$-fixed point, while in $cc(B^n)$ the $O(n)$-fixed point set is homeomorphic to the unit segment $[0, 1]$. These facts give a background to expect a close relationship between the Banach–Mazur compactum $L(n)/O(n)$ and the orbit space $cc(B^n)/O(n)$. The author believes that non of them is homeomorphic to the Hilbert cube. At the moment this is proved only for $L(2)/O(2)$ (see [10]).

In this connection it is interesting to ask the following

Question 4.10. *What is the relationship between the Banach–Mazur compactum $L(n)/O(n)$ and the orbit space $cc(B^n)/O(n)$?*

Yet another interesting representation of the orthogonal group $O(n)$ on the Hilbert cube arises in the following way. There is a well-known Curtis–Schori–West hyperspace theorem (see e.g., [20, Theorem 8.4.5]) asserting that the hyperspace $\exp B^n$ of all nonempty closed subsets $A \subset B^n$ is homeomorphic to the Hilbert cube. It was proved in [12] that $\exp B^n$, endowed with the induced action of the orthogonal group $O(n)$, is an $O(n)$-*AE*. So again, by [6, Theorem 8], the orbit space $(\exp B^n)/O(n)$ is an *AE*. But this time it is indeed homeomorphic to the Hilbert cube (this is an unpublished result of the author). The essential difference of $\exp B^n$ from the two previous actions of $O(n)$ on the Hilbert cubes $cc(B^n)$ and $L(n)$ is that in $\exp B^n$ the K-fixed point set $(\exp B^n)[K]$ *is a Hilbert cube* for every closed subgroup $K \subset O(n)$. Indeed $(\exp B^n)[K] \cong \exp(B^n/K)$, which is a Hilbert cube by the above quoted Curtis–Schori–West hyperspace theorem.

In conclusion we would like to state the following

Conjecture 4.11. *Let a compact Lie group G act on the Hilbert cube Q in such a way that $Q \in G$-AE. Then the orbit space Q/G is homeomorphic to Q iff for each closed subgroup $K \subset G$, the K-fixed point set $Q[K]$ is homeomorphic to Q.*

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